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# Rate of Escape from Nonattracting Chaotic Sets

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Chaotic transient phenomena occur in the vicinity of nonattracting chaotic sets. The rate of escape measures the average length of the transients. There is a conjecture by Eckmann and Ruelle connecting the rate of escape to the Lyapunov exponents and entropy. We prove an inequality that partially supports the conjecture.

**KEY WORDS:** Rate of escape; chaotic transients; Lyapunov exponents; entropy.

## 1. INTRODUCTION

Chaotic transient behavior is often observed in numerical or experimental studies of dynamical systems. This phenomenon is usually due to the presence of a *nonattracting chaotic set.* For almost every initial point near a nonattracting set, the orbit will sooner or later move away from the set and approach some attractor. However, if the set is chaotic, the orbits do not escape immediately. Some of them spend a sufficiently long time in the vicinity of the set before eventually getting out of it. Tracing such an orbit, one thus observes transient chaos.

An interesting problem is to investigate the time period a typical orbit remains in the vicinity of the chaotic set, where it behaves chaotically. This leads to the study of the escape rate.[6,9,12] For a fixed neighborhood $U$, the *escape time* $T(x)$ of a point $x$ is the maximum value $n$ such that the orbit of $x$ stays in $U$ up to time $n-1$. The escape time $T(x)$ is infinity if the orbit never leaves $U$. Let $U_n$ be the set of $x$ with the property $T(x) \geqslant n$. The volume of $U_n$ decays roughly as an exponential function of $n$. The exponential rate is called the *rate of escape* from $U$, or simply the *escape rate.* In general, a bigger escape rate means a shorter transient time.

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A heuristic formula due to Kantz and Grassberger[9] has been used in the literature,[7,3] which connects the escape rate with Lyapunov exponents and dimensions. The formula was reinterpreted by Eckmann and Ruelle[5] and was shown to be rigorous in the case that the nonattracting set is uniformly hyperbolic. Then they went further to propose a conjecture that the formula should be true for nonhyperbolic sets as well. This paper is devoted to proving a partial result on nonhyperbolic sets which supports the conjecture. In the rest of this section we give a detailed review of the Kantz–Grassberger formula and the Eckmann–Ruelle conjecture. Let us begin with the well-understood uniformly hyperbolic case.

Assume that $f$ is a $C^2$ diffeomorphism on a compact manifold. Let $\Lambda$ be an isolated and topologically transitive hyperbolic set. Also assume that $\Lambda$ is not an attractor, that is, the stable manifolds do not fill a neighborhood of $\Lambda$. As a typical example in dimension 2, one may think of the Smale horseshoe. If $U$ is a small neighborhood of $\Lambda$, set

$$\begin{aligned} U_n &= \{x: T(x) \geqslant n\} \\ &= \{x: f^i x \in U, i = 0, 1,..., n-1\} \end{aligned}$$

where $U_1 = U$. Use $m$ to denote the volume on $M$. By ref. 2 the following limit exists:

$$\lim_{n \to \infty} -\frac{1}{n} \log m(U_n)$$

The limit, denoted by $r(\Lambda)$, is independent of $U$ and is given by

$$r(\Lambda) = -\sup \left\{ h(\mu) - \sum \text{positive } \lambda_i(\mu): \right.$$
$$\left. \mu \text{ is an ergodic measure with support in } \Lambda \right\} \tag{1}$$

where $h(\mu)$ is the entropy of $\mu$ and the $\lambda_i(\mu)$ are the Lyapunov exponents. By an inequality of Ruelle (see ref. 5), $h(\mu) - \sum$ positive $\lambda_i(\mu) \leqslant 0$. So $r(\Lambda) \geqslant 0$. Writing in exponential form, we obtain

$$m(U_n) = e^{-nr_n}$$

with $\lim_{n \to \infty} r_n = r(\Lambda)$. Thus $r(\Lambda)$ is the asymptotic rate of decay of the volume $m(U_n)$, that is, the escape rate.

There exists a unique ergodic measure $\mu_0$ assuming the sup in (1),[2] which is a so-called equilibrium state. The support of $\mu_0$, supp($\mu_0$), is the whole of $\Lambda$. Therefore, $\mu_0$ is *the* natural measure which characterizes the transient chaos near $\Lambda$. If each exponent $\lambda_i(\mu_0)$ is of multiplicity $m_i$, and the partial dimension of $\mu_0$ along the corresponding unstable manifolds is $D_i$, then by ref. 10 the entropy $h(\mu_0)$ can be written as

$$h(\mu_0) = \sum_{\lambda_i(\mu_0) > 0} D_i \lambda_i(\mu_0)$$

So from (1)

$$\begin{aligned} r(\Lambda) &= -\Big(h(\mu_0) - \sum_{\lambda_i(\mu_0) > 0} m_i \lambda_i(\mu_0)\Big) \\ &= \sum_{\lambda_i(\mu_0) > 0} (m_i - D_i)\, \lambda_i(\mu_0) \end{aligned} \tag{2}$$

where the latter expression (2) is the renowned Kantz–Grassberger formula.

As is known, chaotic transients are more often observed in the vicinity of nonattracting sets which are not uniformly hyperbolic, or whose hyperbolicity is hard to check. As an example, we consider the Hénon map with certain parameter values. Let the diffeomorphism $f$ on $\mathbf{R}^2$ be given by

$$f(x, y) = (A - x^2 + My, x)$$

$f$ is equivalent to the Hénon map under the linear change of variables $x = AX$, $y = AY/M$.[4] Choose parameters $A = 3.1$ and $M = 0.3$. There is a compact invariant set $\Lambda$ contained in the square $U = (-3, 3) \times (-3, 3)$ (Fig. 1). Almost every point in $U$ leaves the region after a certain number of iterations. But the points sufficiently close to $\Lambda$ exhibit long chaotic transients. The set $\Lambda$ is fractal-like and apparently has one stable direction and one unstable direction. However, as far as we know, its uniform hyperbolicity has not been established yet. In particular, these parameter values do not satisfy the condition in ref. 4 that $\Lambda$ is a hyperbolic set.

Therefore, it is important to study the rate of escape from nonattracting sets which are not hyperbolic. This question was stressed by Eckmann and Ruelle in their survey paper.[5] Let $\Lambda$ be a compact invariant set and $\mu_0$ be an ergodic measure with support in $\Lambda$ such that

$$h(\mu_0) - \sum \text{positive } \lambda_i(\mu_0) \geq h(\mu) - \sum \text{positive } \lambda_i(\mu) \tag{3}$$

for all ergodic $\mu$ in $\Lambda$. In view of the hyperbolic case we discussed before, it is not hard to understand the following conjecture.

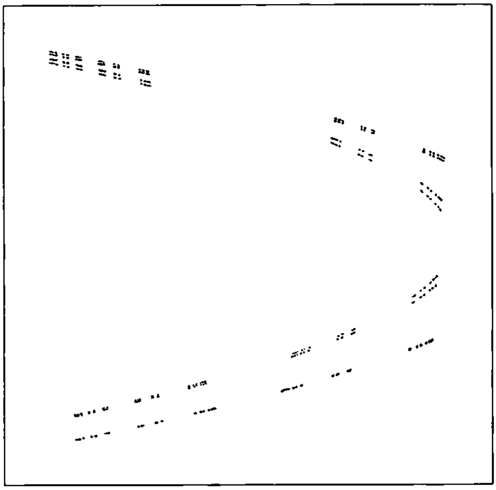

Fig. 1. A chaotic compact invariant set of the map $f(x, y) = (3.1 - x^2 + 0.3y, x)$ in the region $[-3, 3] \times [-3, 3]$. The program *Dynamics* by J. A. Yorke is used to create the picture.

**Conjecture.**[5] The rate of escape from $\Lambda$ is given by

$$r(\Lambda) = -\left(h(\mu_0) - \sum \text{positive } \lambda_i(\mu_0)\right) \tag{4}$$

In other words, $\mu_0$ is a natural measure that describes the transient behavior near $\Lambda$.

Our main effort is to show that if $\mu_0$ satisfies (3), then the rate of escape from any neighborhood $U$ of $\Lambda$ is no greater than $-(h(\mu_0) - \sum$ positive $\lambda_i(\mu_0))$. That is, asymptotically, $m(U_n)$ is at least

$$\exp\left[n\left(h(\mu_0) - \sum \text{positive } \lambda_i(\mu_0)\right)\right]$$

This verifies half of the Eckmann–Ruelle conjecture. The rigorous results in this paper will be restricted to the two-dimensional case. In two dimensions the techniques and notations are much simpler, while the idea can easily be generalized to prove corresponding high-dimensional results.

Although the complete proof of the Eckmann–Ruelle conjecture is still missing, numerical and experimental studies often show that the right-hand side of (2) or (4) is a good approximation to the escape rate when there appears to be an ergodic measure $\mu_0$ satisfying (3) and $\text{supp}(\mu_0) = \Lambda$. If such a natural measure does not exist, however, the escape rate is possibly much smaller than $-(h(\mu) - \sum$ positive $\lambda_i(\mu))$ for any $\mu$. For instance, the fractal basin boundaries provide an interesting class of examples which give rise to transient chaos.[6,1] But usually we have little knowledge of the existence of a natural ergodic measure, except when a basin boundary happens to be hyperbolic (e.g., it is a hyperbolic Julia set). Therefore, in most cases a practical strategy would be as follows. First, check the existence of a natural measure by examining the experimental data. If the result is positive, then it is appropriate to apply formula (4).

## 2. MAIN RESULTS

Let $f$: $M \to M$ be a $C^2$ diffeomorphism on a compact surface $M$ and let $\mu$ be an ergodic probability measure. There are two numbers $\lambda_1(\mu) \geqslant \lambda_2(\mu)$ such that for $\mu$-a.e. $x$

$$\lim_{n \to \infty} \frac{1}{n} \log \|D_x f^n\| = \lambda_1(\mu)$$

and

$$\lim_{n \to \infty} -\frac{1}{n} \log \|D_x f^{-n}\| = \lambda_2(\mu)$$

$\lambda_1(\mu)$ and $\lambda_2(\mu)$ are called the *Lyapunov exponents* of $\mu$.

There are a variety of equivalent definitions of the entropy $h(\mu)$. It is convenient for us to take the one from ref. 8. Denote by $d(\cdot, \cdot)$ the distance on the surface $M$ and by $B(x, r)$ the open ball at $x$ with radius $r$. Let

$$B_n(x, r) = \{ y: f^i y \in B(f^i x, r), i = 0, 1,..., n-1 \}$$

For $r > 0$ and $\delta > 0$, use $N(n, r, \delta)$ to denote the smallest number $N$ such that one can find sets $B_n(x_j, r)$, $j = 1,..., N$, satisfying

$$\mu\left( \bigcup_{j=1}^{N} B_n(x_j, r) \right) \geqslant 1 - \delta$$

Then the *entropy* of $\mu$ is given by

$$h(\mu) = \lim_{r \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(n, r, \delta)$$

This quantity is independent of $\delta$. See the excellent survey in ref. 5 for more about the Lyapunov exponents and entropy.

The support of a measure $\mu$, $\mathrm{supp}(\mu)$, is defined as the smallest closed set $K$ with $\mu(K)=1$. If $\mu$ is an invariant measure, then $\mathrm{supp}(\mu)$ is an invariant set, namely, $f(\mathrm{supp}(\mu))=\mathrm{supp}(\mu)$. If $U$ is open and $U\supset\mathrm{supp}(\mu)$, set

$$U_n=\{x: f^i x\in U,\ i=0, 1,..., n-1\}$$

Denote by $m$ the volume on $M$. ($m$ is actually the area, since $M$ is two dimensional.) The following theorem gives a bound for the rate of escape from $\mathrm{supp}(\mu)$.

**Theorem 1.** Assume

$$\lambda_1(\mu)>0>\lambda_2(\mu) \tag{5}$$

Then for any $U\supset\mathrm{supp}(\mu)$,

$$\liminf_{n\to\infty}\frac{1}{n}\log m(U_n)\geqslant h(\mu)-\lambda_1(\mu) \tag{6}$$

Condition (6) means that such a measure $\mu$ retains in its vicinity a volume of no less than

$$e^{n(h(\mu)-\lambda_1(\mu)-\varepsilon)}$$

by time $n$, where $\varepsilon>0$ can be arbitrarily small. The theorem will be proved in Section 3.

Condition (5) is typically satisfied by a *chaotic* ergodic measure. Let us take a brief look at the other cases. If $0>\lambda_1(\mu)\geqslant\lambda_2(\mu)$ or $\lambda_1(\mu)\geqslant\lambda_2(\mu)>0$, then $\mu$ is supported on a periodic orbit. $\mathrm{supp}(\mu)$ is not chaotic and $h(\mu)=0$. It is easy to calculate the corresponding escape rates:

$$r(\mathrm{supp}(\mu))=0$$

in the first (attracting) case; and

$$r(\mathrm{supp}(\mu))=\lambda_1(\mu)+\lambda_2(\mu)$$

in the second (repelling) case. When $\mu$ is *degenerate*, in the sense that one of the Lyapunov exponents (or both of them) is equal to zero, the entropy $h(\mu)=0$ and $\mathrm{supp}(\mu)$ may or may not be a fractal set. The following can be shown:

$$r(\mathrm{supp}(\mu))=0 \qquad \text{if} \quad \lambda_1(\mu)=0\geqslant\lambda_2(\mu)$$

and

$$\liminf_{n \to \infty} \frac{1}{n} \log m(U_n) \geqslant -\lambda_1(\mu) \qquad \text{if} \quad \lambda_1(\mu) \geqslant 0 = \lambda_2(\mu)$$

We do not write $r(\text{supp}(\mu))$ in the latter case because, as in (6), the limit $\lim_{n \to \infty}$ usually does not exist. The proof of these two formulas will not be given in this paper, and is basically a slight modification of the proof in Section 3.

The entropy $h(\mu)$ is often used as an indicator for the chaoticity of a measure $\mu$. In this sense $\mu$ is chaotic if and only if $h(\mu) > 0$. It is well known (e.g., see ref. 10) that $h(\mu) > 0$ implies $\lambda_1(\mu) > 0 > \lambda_2(\mu)$. Also from ref. 10 we have

$$h(\mu) = D_1 \lambda_1(\mu)$$

where $D_1$ is the Hausdorff dimension of $\mu$ on the unstable manifolds. Thus the right-hand side of (6) can be written as

$$-(1 - D_1)\, \lambda_1(\mu)$$

This form is favored by many authors.[9,7]

In particular, if $\mu$ is a Sinai–Bowen–Ruelle measure,[2] i.e., $\mu$ has absolutely continuous conditional measures on the one-dimensional unstable manifolds, then $\text{supp}(\mu)$ is a chaotic attractor. It is known that $h(\mu) = \lambda_1(\mu)$ (Pesin's formula) and $D_1 = 1$. Condition (6) implies

$$r(\text{supp}(\mu)) = \lim_{n \to \infty} -\frac{1}{n} \log m(U_n) = 0$$

It is compatible with the attractive property of $\text{supp}(\mu)$.

The following is an easy consequence of the theorem.

**Corollary 1.** If $\Lambda$ is a compact invariant set and $U \supset \Lambda$ is open, then

$$\liminf_{n \to \infty} \frac{1}{n} \log m(U_n) \geqslant \sup\{h(\mu) - \lambda_1(\mu) \colon \mu \text{ ergodic}, \text{supp}(\mu) \subset \Lambda \text{ and (5) holds}\} \qquad (7)$$

From the above discussion, the condition that (5) holds can be dropped. If there happens to be an ergodic measure $\mu_0$ with $\text{supp}(\mu_0) = \Lambda$ which achieves the sup in (7), then (7) becomes

$$\liminf_{n \to \infty} \frac{1}{n} \log m(U_n) \geqslant h(\mu_0) - \lambda_1(\mu_0)$$

which confirms one side of the Eckmann–Ruelle conjecture. When $\Lambda$ is not a hyperbolic set, however, it seems difficult to bound $\limsup_{n\to\infty}(1/n) \log m(U_n)$ from above by $h(\mu_0)$ and $\lambda_1(\mu_0)$, even if such a natural measure $\mu_0$ exists. The beautiful formula (1) [or (2)] for a hyperbolic $\Lambda$ is largely due to the fact that the hyperbolicity can be uniformly extended to a neighborhood of $\Lambda$. So the dynamical quantities on $\Lambda$ precisely reflect the dynamics in that neighborhood. In contrast, if $\Lambda$ is not hyperbolic, the dynamics on $\Lambda$ generally does not give a good control on any of its neighborhood, no matter how small. Therefore, the other half of the Eckmann–Ruelle conjecture is probably only true under proper modifications.

## 3. PROOF

We prove the theorem in this section. Several lemmas are required. Let us first introduce the Lyapunov charts.

Let $\mu$ be an ergodic probability measure satisfying (5). For simplicity, write $h$, $\lambda_1$, and $\lambda_2$ for $h(\mu)$, $\lambda_1(\mu)$, and $\lambda_2(\mu)$, respectively. There is a set $\Gamma \subset \mathrm{supp}(\mu)$ with $\mu(\Gamma)=1$, such that a family of coordinate changes can be constructed around the points $x \in \Gamma$ via which $f$ becomes uniformly hyperbolic. These coordinate changes are called the *Lyapunov charts*. We collect some facts that will be used later. See ref. 10 for details.

Let $R(r)$ be the square $(-r, r)\times(-r, r)$ in $\mathbf{R}^2$. For any given $0<\varepsilon \leqslant \min\{\lambda_1, |\lambda_2|\}$ there is a measurable function $l\colon \Gamma \to [1, \infty)$ that varies slowly along orbits

$$1-\varepsilon \leqslant l(fx)/l(x) \leqslant 1+\varepsilon \tag{8}$$

The chart at $x \in \Gamma$ is a square $R(\varepsilon l(x)^{-1})$ together with an embedding

$$\Phi_x\colon R(\varepsilon l(x)^{-1}) \to M$$

satisfying the following properties:

(i) $\Phi_x(0)=x$.

(ii) For a constant $K>1$

$$K^{-1} d(\Phi_x y_1, \Phi_x y_2) \leqslant \|y_1 - y_2\| \leqslant l(x)\, d(\Phi_x y_1, \Phi_x y_2)$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^2$.

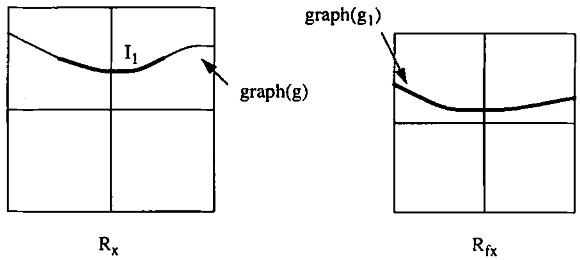


Fig. 2. $\tilde{f}_x$ maps $I_1$ onto graph($g_1$).

Further, if $\tilde{f}_x = \Phi_{fx}^{-1} \circ f \circ \Phi_x$ is the induced map between charts, then:

(iii) We have

$$D\tilde{f}_x|_0 = \begin{pmatrix} \chi_1(x) & 0 \\ 0 & \chi_2(x) \end{pmatrix}$$

where $(1-\varepsilon)\, e^{\lambda_i} \leqslant \chi_i(x) \leqslant (1+\varepsilon)\, e^{\lambda_i}$, $i = 1, 2$.

(iv) $\|D\tilde{f}_x|_{y_1} - D\tilde{f}_x|_{y_2}\| \leqslant l(x)\, \|y_1 - y_2\|$.

Let us denote the chart at $x$ by $R_x$, and write $R_x = R_x^u \times R_x^s$, where $R_x^u = R_x^s = (-\varepsilon l(x)^{-1}, \varepsilon l(x)^{-1})$ are the coordinate axes in $R_x$. The graph of a $C^1$ function $g$: $R_x^u \to R_x^s$, denoted by graph($g$), is thus a smooth curve in $R_x$. The following lemma shows how such a curve gets transformed by $\tilde{f}_x$.

**Lemma 1.** If $g$: $R_x^u \to R_x^s$ is $C^1$ and $|g'| \leqslant 1$, then there is a connected piece $I_1 \subset \text{graph}(g)$ such that $\tilde{f}_x(I_1)$ is a graph of a $C^1$ function $g_1: R_{fx}^u \to R_{fx}^s$ with $|g_1'| \leqslant 1$ (see Fig. 2).

The statement has appeared in many papers; e.g., see ref. 11, p. 124. It results from the fact that $\tilde{f}_x$ is expanding in the $R_x^u$ direction and contracting in the $R_x^s$ direction.

We introduce some notations. If $y \in R_x$, write $y = (y^u, y^s)$. Accordingly, $\tilde{f}_x(y) = (\tilde{f}_x^u(y^u, y^s), \tilde{f}_x^s(y^u, y^s))$, and

$$D\tilde{f}_x = \begin{pmatrix} D\tilde{f}_x^u \\ D\tilde{f}_x^s \end{pmatrix}$$

**Lemma 2.** Assume $g$: $R_x^u \to R_x^s$ is $C^1$ and $|g'| \leqslant 1$. Then

$$(1-\psi_1(\varepsilon))\, e^{\lambda_1} \leqslant \frac{d}{dz} \tilde{f}_x^u(z, g(z)) \leqslant (1+\psi_1(\varepsilon))\, e^{\lambda_1}$$

whenever $(z, g(z)) \in I_1$, where

$$\psi_1(\varepsilon) = \varepsilon + 2\varepsilon/e^{\lambda_1}$$

*Proof.* We have

$$
\begin{aligned}
&\frac{d}{dz}\tilde{f}^u_x(z, g(z)) \\
&\quad = D\tilde{f}^u_x|_{(z,g(z))}\begin{pmatrix}1\\ g'(z)\end{pmatrix} \\
&\quad = (D\tilde{f}^u_x|_{(z,g(z))} - D\tilde{f}^u_x|_{(0,0)})\begin{pmatrix}1\\ g'(z)\end{pmatrix} + D\tilde{f}^u_x|_{(0,0)}\begin{pmatrix}1\\ g'(z)\end{pmatrix} \qquad (9)
\end{aligned}
$$

By (iv),

$$
\begin{aligned}
\|D\tilde{f}^u_x|_{(z,g(z))} - D\tilde{f}^u_x|_{(0,0)}\| &\leqslant l(x)\ \|(z, g(z))\| \\
&\leqslant l(x)\cdot\sqrt{2}\varepsilon l(x)^{-1} = \sqrt{2}\varepsilon
\end{aligned}
$$

$|g'| \leqslant 1$ implies

$$
\left\|\begin{pmatrix}1\\ g'(z)\end{pmatrix}\right\| \leqslant \sqrt{2}
$$

On the other hand,

$$
\begin{aligned}
D\tilde{f}^u_x|_{(0,0)}\begin{pmatrix}1\\ g'(z)\end{pmatrix} &= (\chi_1(x), 0)\begin{pmatrix}1\\ g'(z)\end{pmatrix} \\
&= \chi_1(x)
\end{aligned}
$$

where the first equation is from (iii). Using

$$
(1-\varepsilon)\, e^{\lambda_1} \leqslant \chi_1(x) \leqslant (1+\varepsilon)\, e^{\lambda_1}
$$

and by (9), we have

$$
\begin{aligned}
\frac{d}{dz}\tilde{f}^u_x(z, g(z)) &\leqslant 2\varepsilon + \chi_1(x) \\
&\leqslant \left(1+\varepsilon+\frac{2\varepsilon}{e^{\lambda_1}}\right) e^{\lambda_1}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dz}\tilde{f}^u_x(z, g(z)) &\geqslant -2\varepsilon + \chi_1(x) \\
&\geqslant \left(1-\varepsilon-\frac{2\varepsilon}{e^{\lambda_1}}\right) e^{\lambda_1} \quad \blacksquare
\end{aligned}
$$

Apply Lemma 1 repeatedly. After $n$ iterations, a small piece of curve $I_n \subset \text{graph}(g)$ is mapped by

$$F_x^n = \tilde{f}_{f^{n-1}x} \circ \cdots \circ \tilde{f}_{fx} \circ \tilde{f}_x$$

onto graph$(g_n)$ for some $C^1$ function

$$g_n \colon R^u_{f^n x} \to R^s_{f^n x}$$

Let us calculate the length of $I_n$.

**Lemma 3.** The arclength of $I_n$ satisfies the condition

$$L(I_n) \geqslant 2\varepsilon l(x)^{-1} \cdot (1 + \psi_2(\varepsilon))^{-n} e^{-n\lambda_1}$$

where

$$\psi_2(\varepsilon) = \varepsilon + \psi_1(\varepsilon) + \varepsilon\psi_1(\varepsilon)$$

*Proof.* $I_n$ is the graph of $g$ over some interval $(a, b) \subset R^u_x$. For any $z = z_0 \in (a, b)$, denote

$$z_{k+1} = \tilde{f}^u_{f^k x}(z_k, g_k(z_k)), \qquad k = 0, 1, \ldots, n-1$$

$(g_0 = g)$. Then

$$\begin{aligned} \frac{d}{dz}(F_x^n)^u (z, g(z)) &= \prod_{k=0}^{n-1} \frac{d}{dz} \tilde{f}^u_{f^k x}(z_k, g_k(z_k)) \\ &\leqslant (1 + \psi_1(\varepsilon))^n e^{n\lambda_1} \end{aligned}$$

where the inequality is by Lemma 2. Since

$$\int_a^b \frac{d}{dz}(F_x^n)^u (z, g(z))\, dz = 2\varepsilon l(f^n x)^{-1}$$

and

$$l(f^n x) \leqslant l(x)(1+\varepsilon)^n$$

we have

$$\begin{aligned} b - a &\geqslant 2\varepsilon l(x)^{-1} (1+\varepsilon)^{-n} (1 + \psi_1(\varepsilon))^{-n} e^{-n\lambda_1} \\ &= 2\varepsilon l(x)^{-1} (1 + \psi_2(\varepsilon))^{-n} e^{-n\lambda_1} \end{aligned}$$

Obviously, $L(I_n) \geqslant b - a$; thus the lemma is proved. ∎

By (ii), for any $y \in R_x$,

$$d(\Phi_x y, x) \leqslant K \|y - 0\| < K\sqrt{2}\varepsilon l(x)^{-1} \leqslant \sqrt{2}K\varepsilon$$

that is,

$$\Phi_x(R_x) \subset B(x, \sqrt{2}K\varepsilon) \tag{10}$$

**Lemma 4.** If $x \in \Gamma$, then

$$m(B_n(x, \sqrt{2}K\varepsilon)) \geqslant 4\varepsilon^2 l(x)^{-4} (1 + \psi_2(\varepsilon))^{-n} e^{-n\lambda_1}$$

*Proof.* By (10), $B_n(x, \sqrt{2}K\varepsilon) \supset \Phi_x(\tilde{B}_n(x, \varepsilon))$, where

$$\tilde{B}_n(x, \varepsilon) = \{ y \in R_x \colon F_x^k y \in R_{f^k x}, k = 1,..., n-1 \}$$

For any $v \in R_x^s$, let $J_v$ be the line segment

$$J_v = \{ (y^u, v) \colon y^u \in (-\varepsilon l(x)^{-1}, \varepsilon l(x)^{-1}) \}$$

By Lemma 3, there is a small piece of $J_v$ with length $\geqslant 2\varepsilon l(x)^{-1} (1 + \psi_2(\varepsilon))^{-n} e^{-n\lambda_1}$ that is contained in $\tilde{B}_n(x, \varepsilon)$. Integrating with respect to $v$ over $R_x^s = (-\varepsilon l(x)^{-1}, \varepsilon l(x)^{-1})$, we obtain an area greater than or equal to

$$2\varepsilon l(x)^{-1} \cdot 2\varepsilon l(x)^{-1} (1 + \psi_2(\varepsilon))^{-n} e^{-n\lambda_1} = 4\varepsilon^2 l(x)^{-2} (1 + \psi_2(\varepsilon))^{-n} e^{-n\lambda_1}$$

which is contained in $\tilde{B}_n(x, \varepsilon)$. Applying (ii) yields

$$m(B_n(x, \sqrt{2}K\varepsilon)) \geqslant l(x)^{-2} \cdot \{\text{area of } \tilde{B}_n(x, \varepsilon)\} \geqslant 4\varepsilon^2 l(x)^{-4} (1 + \psi_2(\varepsilon))^{-n} e^{-n\lambda_1} \quad \blacksquare$$

Now we can complete the proof of the theorem. Our technique is similar to that in ref. 2 or ref. 11. Suppose $U$ contains an $r$-neighborhood of supp($\mu$). Choose $\varepsilon < r/\sqrt{2}K$. Then for any $x \in \Gamma$, $B_n(x, \sqrt{2}K\varepsilon) \subset U_n$. Let $\Gamma_l = \{x \in \Gamma \colon l(x) \leqslant l\}$. Then $\lim_{l \to \infty} \Gamma_l = \Gamma$. Fix $l$ so large that $\mu(\Gamma_l) \geqslant 1/2$. Thus if $x \in \Gamma_l$,

$$m(B_n(x, \sqrt{2}K\varepsilon)) \geqslant 4\varepsilon^2 l^{-4} (1 + \psi_2(\varepsilon))^{-n} e^{-n\lambda_1}$$

Let $\{x_j\colon j=1,\dots,J\}$ be a maximal $(n, 2\sqrt{2}K\varepsilon)$-separated set in $\Gamma_l$, i.e., $x_j \in \Gamma_l$,

$$x_j \notin B_n(x_{j'}, 2\sqrt{2}K\varepsilon) \qquad \text{if} \quad j \neq j' \tag{11}$$

and

$$\bigcup_{j=1}^{J} B_n(x_j, 2\sqrt{2}K\varepsilon) \supset \Gamma_l \tag{12}$$

Now, (12) gives

$$J \geqslant N(n, 2\sqrt{2}K\varepsilon, \tfrac{1}{2})$$

But by (11), the sets $\{B_n(x_j, \sqrt{2}K\varepsilon)\colon j=1,\dots,J\}$ are disjoint. So

$$\begin{aligned} m(U_n) &\geqslant m\left(\bigcup_{j=1}^{J} B_n(x_j, \sqrt{2}K\varepsilon)\right) \\ &= \sum_{j=1}^{J} m(B_n(x_j, \sqrt{2}K\varepsilon)) \\ &\geqslant N(n, 2\sqrt{2}K\varepsilon, \tfrac{1}{2}) \cdot 4\varepsilon^2 l^{-4}(1+\psi_2(\varepsilon))^{-n}\, e^{-n\lambda_1} \end{aligned}$$

Therefore,

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \log m(U_n) \\ &\qquad \geqslant \liminf_{n\to\infty} \frac{1}{n} \log N\left(n, 2\sqrt{2}K\varepsilon, \frac{1}{2}\right) - \log(1+\psi_2(\varepsilon)) - \lambda_1 \end{aligned} \tag{13}$$

It is obvious from the definition of $\psi_1$ and $\psi_2$ that

$$\lim_{\varepsilon\to 0} \psi_2(\varepsilon) = 0$$

Taking $\varepsilon \to 0$ in (13) and using the definition of entropy, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log m(U_n) \geqslant h - \lambda_1 \quad \blacksquare$$

**Remark.** Our method can be easily generalized to prove a corresponding multidimensional result, where some of the Lyapunov exponents are allowed to be equal to zero.

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